Molien Series and a Recent Theorem of Kostant

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Outline

Outline of the Talk

- Kostant's recent result

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- New Theoretical Results

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$$\pi(g)\pi(g)^* = \pi(g)^*\pi(g) = I$$

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(π, V) is *irreducible* if the only closed G-invariant subspaces of V are $\{0\}$ and V.

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Within the sets of irreducible representations the unitary ones are fundamental and play an increasing rôle in many branches of Mathematics and Physics.

We can regard the Hilbert space $L^2(S^1)$, the square integrable functions on the circle, with the translation group action as a unitary representation of S^1 due to the fact that every function in $L^2(S^1)$ has a Fourier series decomposition:

$$L^2(S^1) = \sum_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta}.$$

In this case the $e^{in\theta}$'s are homomorphisms from S^1 to $\mathbb{C}^*$, the multiplicative group of $\mathbb{C}$. They are unitary of dimension one and $L^2(S^1)$ decomposes into a discrete sum of irreducible unitary representations. Here the compactness of S^1 is enough to guarantee the above discrete decomposition.

If we were to consider $L^2(\mathbb{R})$ then every function in that space would have a Fourier transform and consequently:

$$L^2(\mathbb{R}) = \int_{\mu\in\mathbb{R}} \mathbb{C}e^{ix\mu}d\mu.$$

Theorem

Let G be a non-compact, linear simple Lie group. Then any non trivial irreducible unitary representation π of G is of infinite dimension.

Unitary Dual

The set of irreducible unitary representations (**The Unitary Dual**) of G denoted by $\hat{G}$ is a fundamental tool to understand the actions of G. For G compact $\hat{G}$ is essentially determined. Barbasch has treated the classical complex groups. However the following cases are still not resolved:

Type A: $SU(p, q)$ for $(p, q > 2)$
Type B: $SO(p, q)$ for $(p, q \geq 3)$
Type C: $Sp(p, q)$ for $(p, q \geq 2)$
Type D: $SO(p, q)$ for $(p, q \geq 3)$, $SO^*(2n)$ for $(n \geq 4)$
Type F_4: $F_4(\mathbb{C})$, $F_4(split)$
Type E_6: $E_6(\mathbb{C})$, $E_6(split)$, $E_6(Hermitian)$, $E_6(quaternionic)$
Type E_7: $E_7(\mathbb{C})$ and all real non-compact forms
Type E_8: $E_8(\mathbb{C})$ and all real non-compact forms

Computing the Unitary Dual : $\hat{G}$

Assume G real reductive. Works of Harish-Chandra, Vogan, Barbash and many others relate $\hat{G}$ to $\hat{K}$ where K is a maximal compact subgroup of G.

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Here the fundamental object is the (g, K)-module, a vector space equipped with two compatible actions on $\mathfrak{g}$, the complexification of the Lie algebra of G and K. One is usually interested in Harish-Chandra modules that is (g, K)-modules that have finite multiplicities as representations of K.

Computing the Unitary Dual : $\hat{G}$

• **The Atlas of Lie Groups and Representations**: NSF funded, directed by Jeff Adams (University of Maryland College Park): A group of mathematicians working on theoretical and algorithmic problems to produce a software system that might compute $\hat{E}_{8(8)}$ and hopefully point to a general theorem. Recent works by Vogan, Adams, Yee, Trapa, and van Leeuwen provide a potential algorithm. This will be the subject of an NSF-funded Regional Conference that I am organizing in July 2012 at the University of Massachusetts Boston featuring 10 one-hour lectures by Vogan.

• An attempt to determine irreducible (g, K)-modules (up to infinitesimal equivalence) by the action of $\mathfrak{U}(\mathfrak{g})^K$, the centralizer of the complexified K in the enveloping algebra of $\mathfrak{g}$, on any K-primary component.
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• $\mathfrak{U}(\mathfrak{g})^K$ acts as a one dimensional character and the action is known once the values of the character on the generators of $\mathfrak{U}(\mathfrak{g})^K$ are known.

Outline

1. Outline
2. Preliminaries
3. Generators for $\mathfrak{U}(\mathfrak{g})^K$
4. Jackson-N Algorithms
5. Examples

Previous Works

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Complete Results for: $SU(2,2), SU(n,1), SO(n,1)$. (Very few cases indeed)

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K: the subgroup of the adjoint group of $\mathfrak{g}$ corresponding to $\mathfrak{k}$.
Define a filtration

$$\mathfrak{U}(\mathfrak{g}) = \bigcup_{i=0}^{\infty} (\mathfrak{U}(\mathfrak{g}))_i$$

where $(\mathfrak{U}(\mathfrak{g}))_i$ is the span of all j-fold products of elements of $\mathfrak{g}$ for $j \leq i$.

By the Poincaré-Birkhoff-Witt theorem, the associated graded algebra with respect to this filtration is the symmetric algebra $S(\mathfrak{g})$. This is canonically isomorphic to the algebra of polynomial functions $\mathbb{C}[\mathfrak{g}^*]$, and since $\mathfrak{g}$ is reductive it is self-dual and we can identify $S(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$. In particular, if we can find generators for $\mathbb{C}[\mathfrak{g}]^K$ then any set of liftings of these generators to $\mathfrak{U}(\mathfrak{g})$ will generate $\mathfrak{U}(\mathfrak{g})^K$.

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By the way it is not difficult to construct a linear basis of $\mathfrak{U}(\mathfrak{g})^K$. The problem has to do with its ring structure.

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Journal of Algebra Volume 313, Issue 1, 1 July 2007, Pages 252-267
Special Issue in Honor of Ernest Vinberg.

Theorem (Kostant 2007)

Generators of $\mathfrak{U}(\mathfrak{g})^K$ are obtained from lifting generators of $S(\mathfrak{g})^K$.

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Generators of $\mathfrak{U}(\mathfrak{g})^K$ are obtained from lifting generators of $S(\mathfrak{g})^K$.

$$S(\mathfrak{g})^K = S(\mathfrak{g})^K_r \text{ where } r = \binom{2\dim\mathfrak{g}}{2}\dim\mathfrak{p}$$

$S(\mathfrak{g})^K_r$ the subalgebra of $S(\mathfrak{g})^K$ defined by K-invariant polynomials of degree at most r.

Our naive approach to implement Kostant's method for $SL_3(\mathbb{R})$ leads to matrices with 10^{32} matrices.

Outline

A Molien series for $\mathfrak{U}(\mathfrak{g})^K$

We describe a method by which Kostant's algorithm can be significantly accelerated by exploiting the Kostant-Rallis theorem via a certain homomorphism from $\mathfrak{U}(\mathfrak{g})^K$ to the ring of regular functions on the nilpotent cone in $\mathfrak{p}$. The situation is analogous to that in the invariant theory of finite groups, where the Molien series is used to accelerate the algorithm suggested by Noether's degree bound.

Harm Derksen and Gregor Kemper:*Computational invariant theory, Invariant Theory and Algebraic Transformation Groups, I*, Springer-Verlag, Berlin, 2002, Encyclopaedia of Mathematical Sciences, 130.

A Molien series for $\mathfrak{U}(\mathfrak{g})^K$

Various structural and algebraic considerations yield:

$$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{k}] \otimes \mathbb{C}[\mathfrak{p}] \simeq \mathbb{C}[\mathcal{N}_{\mathfrak{k}}] \otimes \mathbb{C}[\mathfrak{k}]^K \otimes \mathbb{C}[\mathcal{N}_{\mathfrak{p}}] \otimes \mathbb{C}[\mathfrak{p}]^K$$

where $\mathcal{N}_{\mathfrak{k}}$ and $\mathcal{N}_{\mathfrak{p}}$ are the cone of nilpotent elements of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Hence

$$\mathbb{C}[\mathfrak{g}]^K \simeq (\mathbb{C}[\mathcal{N}_{\mathfrak{k}}] \otimes \mathbb{C}[\mathcal{N}_{\mathfrak{p}}])^K \otimes \mathbb{C}[\mathfrak{k}]^K \otimes \mathbb{C}[\mathfrak{p}]^K.$$

This is not an algebra isomorphism but we know any system of generators for the algebra on right hand side will lift to a system of generators for the left.

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[Proposition 10, Proposition 11, Theorem 6, Theorem 9, Theorem 14, Theorem 15, Theorem 17, Remark 21]

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Theorem (Jackson, N)

$\mathcal{A} \simeq \mathbb{C}[\mathcal{N}_{\mathfrak{p}}]^{K^e}$, where e is a regular nilpotent element of $\mathfrak{k}$ and K^e, the isotropy group of e in K.

A Molien series for $\mathfrak{U}(\mathfrak{g})^K$

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This is a grading transferable to $\mathcal{A}$ via the above isomorphism. Define a *Molien series* for $\mathfrak{U}(\mathfrak{g})^K$ as follows:

$$M(t) = \sum_{i=0}^{\infty} (\dim \mathcal{A}_i) t^i.$$

We will see that $\dim \mathcal{A}_i$ is finite.

Computing $M(t)$

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Computing the Molien series can be reduced to computation of M-invariants on $\mathfrak{k}$ together with a Gröbner basis calculation. Since M is frequently the product of a finite group and a torus, this often reduces the computation to familiar algorithms from the invariant theory of finite groups and integer programming.

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Theorem (Jackson, N)

The formal power series $N(t) = \sum_{i=0}^{\infty} (\dim \mathbb{C}[\mathcal{N}_{\mathfrak{k}}]_i^M)\, t^{2i}$ *coincides with the Molien series* $M(t)$ *. In particular, the coefficients of* $M(t)$ *are finite.*

Computing $M(t)$

Let k be the rank of $\mathfrak{k}$ and $u_1, \ldots, u_k$ a system of homogeneous generators for $\mathbb{C}[\mathfrak{k}]^K$ of degrees $d_1, \ldots, d_k$ respectively [Goodman-Wallach]. Then $(u_1, \ldots, u_k)$ is a regular sequence in $\mathbb{C}[\mathfrak{k}]$. In other words codim $\mathcal{N}_{\mathfrak{k}}$ in $\mathfrak{k}$ is k.

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$P(t)$ can be computed either by finding generators for the algebra $\mathbb{C}[\mathfrak{k}]^M$ or, when M is abelian, by invariant integration over a compact real form of M. The former is always possible using standard algorithms (e.g. Derksen's algorithm) or by ad hoc methods, and experience shows that these methods often terminate much more quickly on $\mathbb{C}[\mathfrak{k}]^M$ than on the original invariant ring $\mathbb{C}[\mathfrak{g}]^K$.

Computing $M(t)$

Theorem (Jackson, N)

$$M(t) = P(t^2) \prod_{i=1}^{k} (1 - t^{2d_i}).$$

Computing $M(t)$

M, the cenralizer of $\mathfrak{a}$ in K, is reductive; hence it has a compact real form $M_{\mathbb{R}}$.
Let $x_1, \dots, x_n$ be a basis for $\mathfrak{k}^*$ consisting of eigenvectors for $M_{\mathbb{R}}$ corresponding to characters $\chi_1, \dots, \chi_n$ respectively. We have a natural action of $M_{\mathbb{R}}$ on the formal power series ring $\mathbb{C}[[x_1, \dots, x_n]]$ in which the monomial $x_{i_1} \dots x_{i_k}$ transforms according to the character $\chi_{i_1} \dots \chi_{i_k}$.When M is abelian we can define $R(x_1, \dots, x_n)$ by the formula

$$R(x_1, \dots, x_n) = \int_{m \in M_{\mathbb{R}}} \prod_{i=1}^{n} \frac{1}{1 - m x_i}\, dm.$$

Then $P(t) = R(t, \dots, t)$.

Computing $M(t)$

If the pair (G, K) corresponds to a split real group, then M is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^l$, where l is the rank of G. In particular, M is abelian, and the invariant integral described in the previous section collapses to a finite sum. Since M is a 2-group, we see that x_i^2 is M-invariant for all i. Now let $\mathcal{S}$ denote the set of all M-invariant square-free monomials in $x_1, \ldots, x_n$. Then

$$R(x_1, \ldots, x_n) = \frac{\sum_{\mu \in \mathcal{S}} \mu}{\prod_{i=1}^{n}(1 - x_i^2)}.$$

Computing $M(t)$ for $SL_{\mathbb{R}}(n)$

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For $i < j$ put $m_{i,j}$ be the diagonal matrix with -1's in the ith and jth diagonal positions and ones elsewhere, so that M is generated by the $m_{i,j}$.

Let $\mu = \prod_{i<j} k_{i,j}^{d_{i,j}}$ be any monomial on $\mathfrak{k}$. Define a graph $\Gamma(\mu)$ having vertices $\{v_1, \ldots, v_n\}$ and having exactly $d_{i,j}$ edges connecting v_i with v_j.

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μ can be reconstructed from $\Gamma(\mu)$.

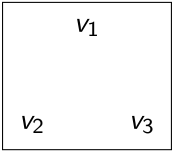
Define $e_i(\mu)$ the degree of v_i in $\Gamma(\mu)$. One has

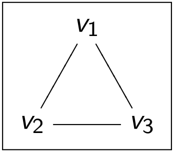
$$m_{i,j}\mu = (-1)^{e_i(\mu)+e_j(\mu)}\mu.$$

μ is square-free if and only if its graph has no multiple edges, and it is M-invariant if and only if the vertex degrees of $\Gamma(\mu)$ are either all even or all odd.

$M(t)$ for $SL_{\mathbb{R}}(3)$

The square-free invariant monomials correspond to the graphs:





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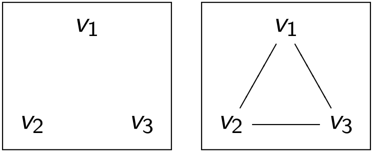

$$P(t) = \frac{1 + t^3}{(1 - t^2)^3}.$$

Since $k = 1$ and $d_1 = 2$,

$$M(t) = \frac{1 + t^6}{(1 - t^4)^2}.$$

$M(t)$ for $SL_{\mathbb{R}}(4)$

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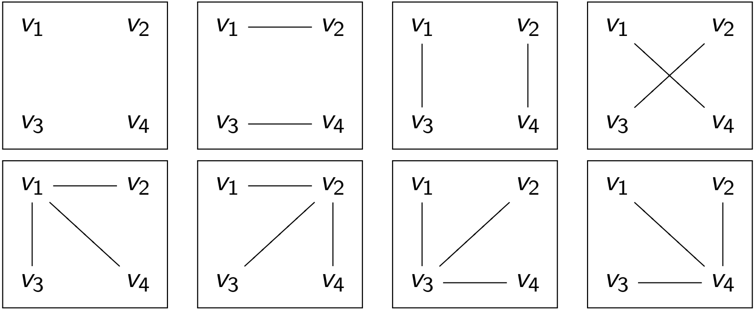

$M(t)$ for $SL_{\mathbb{R}}(4)$

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$$P(t) = \frac{1 + 3t^2 + 8t^3 + 3t^4 + t^6}{(1 - t^2)^6}.$$

Here $k = 2$, $d_1 = 2$, and $d_2 = 2$, so we obtain

$$M(t) = \frac{1 + 3t^4 + 8t^6 + 3t^8 + t^{12}}{(1 - t^4)^4}.$$

$M(t)$ for $SL_{\mathbb{R}}(4)$

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$$M(t) = \frac{1 + 3t^4 + 8t^6 + 3t^8 + t^{12}}{(1 - t^4)^4}.$$

This technique is valid for $SL_{\mathbb{R}}(n)$.

Algorithms

If one knows the Molien series $M(t)$ then one has an algorithm to test whether a given collection of K-invariants $f_1, \ldots, f_n \in \mathbb{C}[\mathfrak{g}]$ (together with the usual generators for $\mathbb{C}[\mathfrak{k}]^K$ and $\mathbb{C}[\mathfrak{p}]^K$) generate $\mathbb{C}[\mathfrak{g}]^K$. Let $\pi : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathcal{N}(\mathfrak{k})] \otimes \mathbb{C}[\mathcal{N}(\mathfrak{p})]$ be the projection, and let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$ generated by $\pi(f_1), \ldots, \pi(f_n)$.

Algorithms

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$Q(t)$ can be computed by a standard extension of Buchberger's algorithm; this extension is implemented in many computer algebra systems, including Macaulay2 which we used.

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We have two ways of manufacturing elements of $\mathbb{C}[\mathfrak{g}]^K$.

Method I: Linear Algebra

Method I is general and produces a basis for $\mathbb{C}[\mathfrak{g}]_d^K$ by computing the Kernel of a matrix. It is suitable for implementation on a computer algebra system. If $M(t)$ is known, this leads immediately to an algorithm which computes generators for $\mathbb{C}[\mathfrak{g}]^K$: starting with $i = 0$, we increment i until a basis for $\sum_{d=0}^{i} \mathbb{C}[\mathfrak{g}]_d^K$ gives $Q(t) = M(t)$.

Method II: Trace Forms

Method II is much faster than Method I. But there is no guarantee that $\mathbb{C}[\mathfrak{g}]^K$ will be generated by trace forms [$Sp_4(\mathbb{R})$ the center of $\mathfrak{k}$ acts tracelessly on any $\mathfrak{g}$-module].

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In general, one can always decompose $\mathfrak{g}$ as a sum of irreducible K-modules:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

for some m. For $1 \leq i \leq m$ let π_i denote the K-equivariant projection from $\mathfrak{g}$ to $\mathfrak{g}_i$. Passing to a representation V of $\mathfrak{g}$, we can regard each $\mathfrak{g}_i$ as a space of matrices on which K acts by conjugation. Now for any sequence $i_1, \ldots, i_d$ with $1 \leq i_j \leq m$, define a function $f_{V,i_1,\ldots,i_d} : \mathfrak{g} \to \mathbb{C}$ by the formula

$$f_{V,i_1,\ldots,i_d}(x) = trace_V(\pi_{i_1}(x) \ldots \pi_{i_d}(x)).$$

Evidently $f_{V,i_1,\ldots,i_d}$ is a polynomial of degree d, and by construction it lies in $\mathbb{C}[\mathfrak{g}]^K$.

Outline

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Now let V denote the standard representation of $\mathfrak{sl}_3$. Using the algorithms discussed earlier we show that:

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generate $\mathbb{C}[\mathfrak{g}]^K$.

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generate $\mathbb{C}[\mathfrak{g}]^K$.
In other words, letting A and S be the antisymmetric and symmetric parts, respectively, of a generic element $x \in \mathfrak{g}$, liftings of the polynomial functions

$$\{tr(A^2), tr(S^2), tr(A^2S), tr(S^3), tr((AS)^2), tr(ASA^2S^2)\}$$

generate $\mathfrak{U}(\mathfrak{g})^K$.

$SL_{\mathbb{R}}(4)$

Letting V denote the standard representation of $\mathfrak{sl}_4$, we can check that $\mathfrak{U}(\mathfrak{g})^K$ is generated by liftings of trace forms on V of degree nine or less.

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$$x = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

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$$x = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then define

$$\pi_1(x) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \pi_2(x) = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \quad \pi_3(x) = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \quad \pi_4(x) = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

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One checks that the trace forms

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generate $\mathbb{C}[\mathfrak{g}]^K$,
whence liftings of the polynomial functions

$$\{tr(A), tr(D), tr(A^2), tr(D^2), tr(BC), tr(ABC), tr(BDC), tr(ABDC), tr((BC)^2), tr(ABDCBC)\}.$$

generate $\mathfrak{U}(\mathfrak{g})^K$. (This agrees with Zhu's result.)

$G_{2(2)}$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ be the short and long simple roots, respectively. Let X_α be the element of a Chevalley basis corresponding to the root α, and put $H_\alpha = [X_\alpha, X_{-\alpha}]$. Conjugating by an inner automorphism if necessary, we may take

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$$\mathfrak{k} = span(X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1}, X_{3\alpha_1+2\alpha_2}, X_{-3\alpha_1-2\alpha_2}, H_{3\alpha_1+2\alpha_2})$$
$$\mathfrak{p} = span(X_{\alpha_2}, X_{-\alpha_2}, X_{\alpha_1+\alpha_2}, X_{-\alpha_1-\alpha_2}, X_{2\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}, X_{3\alpha_1+\alpha_2}, X_{-3\alpha_1-\alpha_2}).$$

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$$M(t) = \frac{1 + 3t^4 + 8t^6 + 3t^8 + t^{12}}{(1 - t^4)^4}$$

$G_{2(2)}$

Put

$$\mathfrak{g}_1 = span(X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1})$$
$$\mathfrak{g}_2 = span(X_{3\alpha_1+2\alpha_1}, X_{-3\alpha_1-2\alpha_2}, H_{3\alpha_1+2\alpha_2})$$
$$\mathfrak{g}_3 = \mathfrak{p}$$

so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ is the decomposition of $\mathfrak{g}$ into irreducible K-modules.

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Let V be the seven-dimensional irreducible representation of $\mathfrak{g}$ induced from the embedding of $\mathfrak{g}_2$ into $\mathfrak{so}_7$. [Fulton & Harris: A First Course in Representation Theory Page 357]

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Let V be the seven-dimensional irreducible representation of $\mathfrak{g}$ induced from the embedding of $\mathfrak{g}_2$ into $\mathfrak{so}_7$. [Fulton & Harris: A First Course in Representation Theory Page 357]
One finds that the trace forms

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$$f_{V,1,1,3,3,3,3}, f_{V,2,1,3,1,1,3}, f_{V,3,3,3,3,3,3}, f_{V,1,3,1,3,3,2,3}, f_{V,1,3,2,3,3,3,3,3},$$
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generate $\mathbb{C}[\mathfrak{g}]^K$.

Todor Milev (Charles University) currently visiting UmassBoston is developing software which among other things compute matrices for representations using information from Littelmann paths. See $http : //vectorpartition.sourceforge.net/$.

THANK YOU